# SEMANTIC WEB BASICS IN LOGICAL CONSIDERATION

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# Outline

1. Tell the name for *description of concepts and relations between them*.

2. Concept graphs and syntactical relations.

3. Syntactical ambiguity of statements in the first-order logic.

4. The decomposability problem.

5. How classical results can be adopted in a modern field of research.

# A quest for a constructive formulation

?

Database Schema

Deductive DB

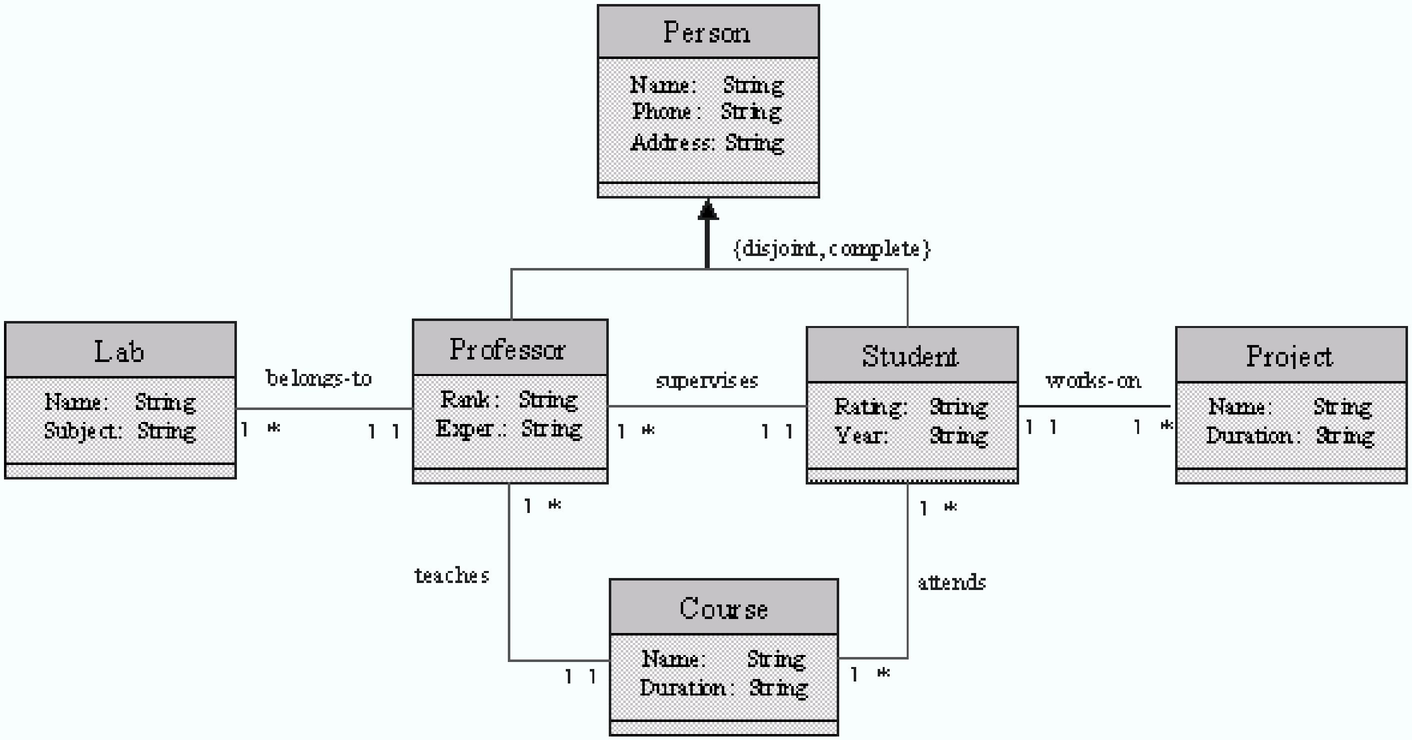
Declarative KB

Ontology

Terminological Database

Logic program

# An Example




$$\forall x(Professor(x) \vee Student(x) \longrightarrow Person(x))$$

$$\forall x(Person(x) \longrightarrow Professor(x) \vee Student(x))$$

$$\forall x \forall y(Supervises(x,y) \longrightarrow Professor(x) \wedge Student(x))$$

$$\forall x \exists y(Professor(x) \longrightarrow Name(y,x))$$

$$\forall x \forall y(Works-on(x,y) \longrightarrow Student(x) \wedge Project(y))$$

$$\forall x \exists y(Project(x) \longrightarrow Duration(y,x))$$

$\ldots$

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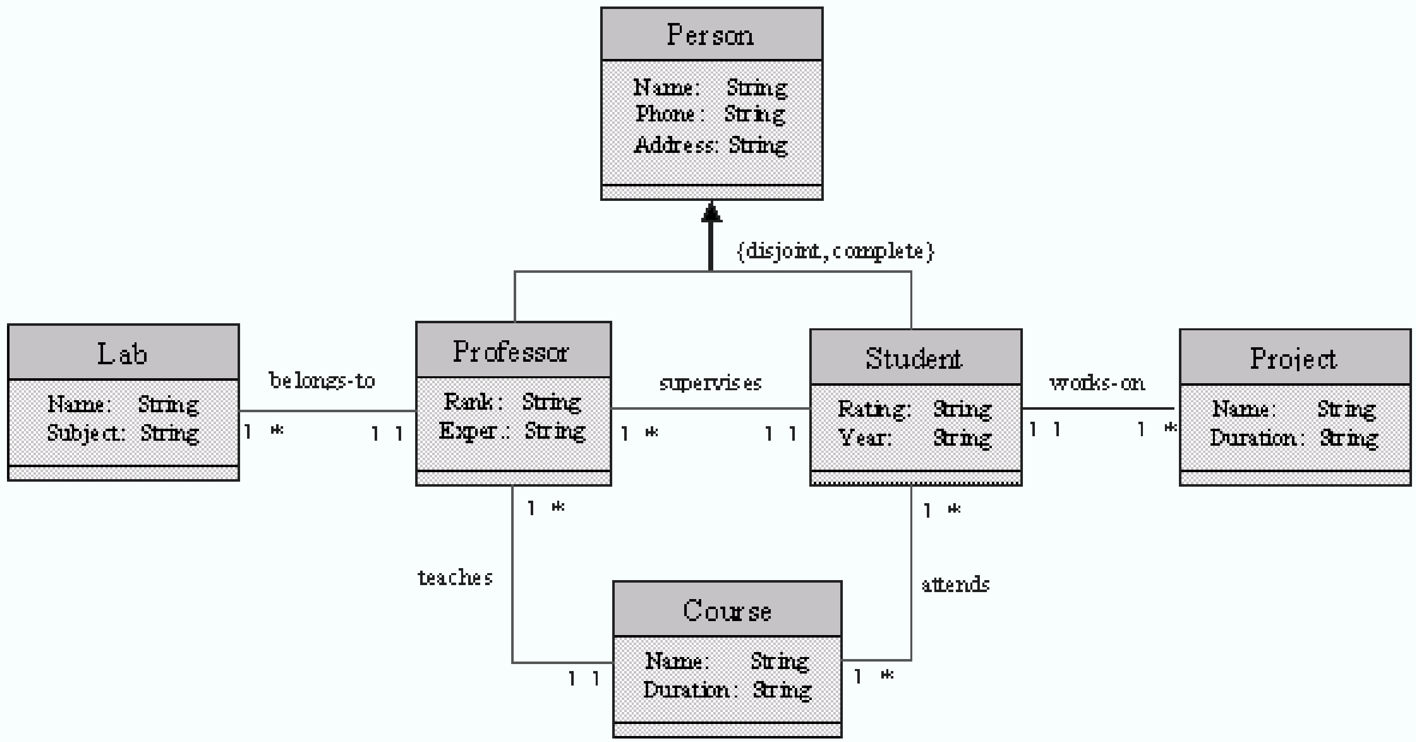
# What syntactical ambiguity can be




$$\forall x \forall y \exists z (\neg Person(x) \vee Professor(x) \vee Student(x) \wedge \neg Project(y) \vee Duration(z, y))$$

$$\forall x \forall z \exists y (\neg Professor(x) \vee Name(y, x) \wedge ToBe(z) \vee \neg ToBe(z))$$

...

Having a set of FOL statements, determine, whether it can be represented as a union of two (or more) sets of statements with non-intersecting signatures.

**Definition 1** *Consider a signature* $\Sigma$ *and a theory* $\mathcal{T}$ *in this signature.*
$\mathcal{T}$ *is called* **decomposable**, *if the signature can be represented as a disjunctive union* $\Sigma = \Sigma_1 \cup \Sigma_2,\ \Sigma_1 \cap \Sigma_2 = \varnothing$, *such that there is a system of axioms* $S = S_1 \cup S_2$, *in which sentences from* $S_i,\ i = 1, 2$ *contain symbols only from* $\Sigma_i$.
*We denote a decomposition of theory as* $\mathcal{T} = S_1 \otimes S_2$, *and decomposition of signature as* $\Sigma = \Sigma_1 \coprod \Sigma_2$.

**Problem 1** *Consider a theory* $\mathcal{T}$ *in a signature* $\Sigma$, *defined by some set of axioms* $\Phi$ *in the signature* $\Sigma$. *How to determine, having the set* $\Phi$, *whether* $\mathcal{T}$ *is decomposable?*

**Problem 2** *Consider a finitely axiomatizable theory* $\mathcal{T}$ *in a signature* $\Sigma$, *defined by some set of axioms* $\Phi$ *in the signature* $\Sigma$. *How to determine, having the set* $\Phi$, *whether* $\mathcal{T}$ *is decomposable?*

**Reformulation** Given an alphabet $\Sigma$, a set of expressions $T_\Sigma$ and a set of [equivalent] transformations on them, infer, whether $T_\Sigma$ is decomposable.

# Applications

1. Knowledge modularization, identification of independent parts of knowledge.

2. Reasoning over large knowledge bases, ontologies.

3. Distributed execution of logical operations on KBs, e.g., consistency cheking.

4. Automatic theorem proving.

5. Program decomposition.

**Craig's interpolation theorem** Let $\varphi$ and $\psi$ be sentences for which $\varphi \vdash \psi$. Then, there is a sentence $\theta$, such that:

1. $\varphi \vdash \theta$ and $\theta \vdash \psi$
2. each symbol in $\theta$ is common for $\varphi$ and $\psi$.

Basing on this classical theorem, we have **proven the following:**

if a theory is decomposable, then

- the corresponding signature decomposition is unique;
- the theory decomposition is unique up to equality formulas.

**Other results:**

1. Main steps to reduce a theory to such *form that uniquely defines this decomposition*; also - the syntactically unambiguous form.
2. Finitely axiomatizable theory: the decomposability problem is reduced to *minimization of a weight of system of axioms*.
3. The decomposition property of theories is stable with respect to simplification procedures, such as *elimination of functional symbols* and *skolemization*.

# Main steps to decomposition

**Remark 1** *If a theory $\mathcal{T}$ in a signature $\Sigma$ can be defined by a system of axioms, which uses only a part of signature symbols $\Sigma' \subset \Sigma$, then this theory is decomposable. The decomposition components in this case are: a theory with the signature $\Sigma'$ and theories with signatures from $\Sigma \backslash \Sigma'$ defined by sets of tautological sentences.*

**Definition 2** *Consider a signature $\Sigma$ and a theory $\mathcal{T}$ in this signature.*
*We call $\mathcal{T}$* **reducible**, *if there exists a subset $\Sigma' \subset \Sigma$ of the signature $\Sigma$ and a system of axioms $S$ for $\mathcal{T}$, which contains symbols only from $\Sigma'$. Thus, $\mathcal{T}$ is* **reduced to the theory** *$\mathcal{T}'$ in the lesser signature $\Sigma'$.*
*If any system of axioms of $\mathcal{T}$ contains all signature symbols of $\Sigma$, then $\mathcal{T}$ is* **irreducible**.
*Let us call* **valid** *those symbols of $\Sigma$ that can not be eliminated from any system of axioms of $\mathcal{T}$.*

**Proposition 1** *Consider a signature extension $\Sigma' \subseteq \Sigma$ and a theory $\mathcal{P}$ in $\Sigma'$. Take a sentence $\varphi$ of the signature $\Sigma$.*
*If $\varphi$ follows from $\mathcal{P}$, then there* **exists a sentence** *$\theta \in \mathcal{P}$, such that:*

- *$\theta$ includes only those symbols of $\Sigma'$, that are present in $\varphi$;*
- *$\varphi$ follows from $\theta : \mathcal{P} \vdash \theta, \;\; \theta \vdash \varphi$.*

**Proposition 2** *Let $\mathcal{T}$ be a theory in a signature $\Sigma$. Consider a set of valid symbols $\Sigma'$ of the signature $\Sigma$: $\Sigma' \subset \Sigma$. Then $\mathcal{T}$ can be defined by a system of axioms in the signature $\Sigma'$. Besides, such a system of axioms defines an irredicible theory.*

# Main steps to decomposition

**Proposition 3** *Consider a signature decomposition* $\Sigma = \Sigma_1 \coprod \Sigma_2$ *and theories* $\mathcal{P}$,$\mathcal{Q}$ *with the signatures* $\Sigma_1$, $\Sigma_2$ *respectively. Consider a sentence* $\varphi$ *in the signature* $\Sigma$.
*If* $\varphi$ *follows from the union of the theories* $\mathcal{P}$, $\mathcal{Q} \ \vdash \varphi$, *then there* **exist sentences** $\theta \in \mathcal{P}$ **and** $\phi \in \mathcal{Q}$, *such that* $\mathcal{P} \vdash \theta$, $\ \mathcal{Q} \vdash \phi$ *and* $\theta, \phi \vdash \varphi$. *Besides,* $\theta$ *includes only those symbols of* $\Sigma_1$ *that are present in* $\varphi$. *Correspondingly,* $\phi$ *contains only those symbols of* $\Sigma_2$ *that are present in* $\varphi$.

**Definition 3** *Consider a theory* $\mathcal{T}$ *and a sentence* $\varphi \in \mathcal{T}$.
$\varphi$ *is* **decomposable in the theory** $\mathcal{T}$, *if there exist sentences* $\theta \in \mathcal{T}, \psi \in \mathcal{T}$, *such that* $\theta, \psi$ *contain symbols only from* $\varphi$ *and do not have common signature symbols, neither of them is an equality formula, and* $\theta, \psi \vdash \varphi$. *We call* $\theta$ *and* $\psi$ *the* **decomposition components** *for the sentence* $\varphi$.
*If there are no such* $\theta$ *and* $\psi$, *then* $\varphi$ *is called* **non-decomposable in the theory** $\mathcal{T}$.

**Proposition 4** *Every theory* $\mathcal{T}$ *has a system of axioms that consists of irreducible non-decomposable sentences.*

The main auxiliary result:

**Lemma 1** *For any non-trivial decomposition* $\mathcal{T}=\mathcal{S}_1 \otimes \mathcal{S}_2$ *in a product of theories with signatures* $\Sigma = \Sigma_1 \coprod \Sigma_2$ $(\Sigma_1 \neq \varnothing \neq \Sigma_2)$, *every non-decomposable sentence* $\varphi \in \mathcal{T}$ *that contains signature symbols, follows only from* $\langle \mathcal{S}_1, \mathcal{T}^{\#} \rangle$ *or only from* $\langle \mathcal{S}_2, \mathcal{T}^{\#} \rangle$.

*If, additionally,* $\varphi$ *is irreducible, then it is contained either in the theory* $\langle \mathcal{S}_1, \mathcal{T}^{\#} \rangle$ *of the signature* $\Sigma_1$, *or in the theory* $\langle \mathcal{S}_2, \mathcal{T}^{\#} \rangle$ *of* $\Sigma_2$. *In particular, it contains symbols only from* $\Sigma_1$ *or only from* $\Sigma_2$.

## How minimization can help

For each formula $\psi$ in a signature $\Sigma$ it is possible to consider the set of symbols $supp(\psi)$ used in this formula. Denote the number of symbols by $np(\psi) = \#(supp(\psi))$.

Let us define a *weight* of a formula $\psi$ as an integer

$$w(\psi) = 3^{np(\psi)} = 3^{\#(supp(\psi))}.$$

**Remark 2** *The number 3 is chosen for the inequality $3^{n+m} > 3^n + 3^m$ to hold for any integers $n$, $m$ (in particular, for the total weight of decomposition fragments of a formula to be less, than the weight of the formula itself).*

**Definition 4** *Let us call the sum of the weights of axioms of a system $\Psi$*

$$w(\Psi) = \sum_{\psi \in \Psi} w(\psi)$$

*as the* **weight of a system** $\Psi$*.*
*Let us call a system $\Psi$ of axioms* **minimal**, *if it has a minimal weight.*

As the weights are represented by integers, it is always possible to choose a system $\Psi$ of axioms of a minimal weight (from known systems of axioms). We will further assume that the system $\Psi$ has a minimal weight. The following proposition explains, how such a system can be used in solving the decomposability problem.

**Proposition 5** *Let $\Psi$ be a minimal system of axioms for a theory $\mathcal{T}$. Then it consists of non-decomposable irreducible sentences of $\mathcal{T}$.*

# Perspectives

**We have:**

1. a (non-constructive) decomposability criterion;
2. uniqueness of decomposition, which justifies search for an algorithm;
3. the minimization task for systems of axioms as a reduction of the decomposability problem;
4. elimination of functional symbols and skolemization do not disturb decomposition components.

**What next:**

1. decidability issues;
2. decomposition algos for concrete cases;
3. expand the framework onto non-classical logics and relative decomposability (when *some* common symbols between components are allowed).

Thank you very much for attention